Warszawa, December 19, 2022

Referee report on a doctoral thesis in Philosophy 'Intuitionistic logic versus paraconsistent logic' by doctor Mariusz Stopa.

I recommend that the thesis of doctor Mariusz Stopa be accepted to be defended by the candidate in front of a Jury.

1 General comments

Even if the thesis does concern logic and category theory, my main fields of study and research in mathematics, it is submitted in philosophy and not in mathematics.

The thesis contains 123 pages. It consists of Introduction, three chapters, and Bibliography. The main contribution of the thesis is of philosophical rather than mathematical nature. Certainly erudition of the candidate plays here a more important role. The main thesis concerns equivalence of two notions of a(n elementary) topos and cotopos. As the 'sameness' relation has many incarnations, before I will comment on the thesis itself, I will make some general comments that, I think, should precede it.

2 Properties, structures, equivalences

There is no good general definition of a structure, even if it is clear that we need to use one in countless situations. Fortunately, it is not so difficult in practice to formulate a reasonable (local) definition of a structure and equivalence of structures so that we can check whether some (kinds of) structures are the same (equivalent) or not. To make the thing even worse, for some structures there are more than one notion of 'being equivalent'. For example, topological spaces might be considered 'equivalent' iff they are isomorphic or iff they are homotopy equivalent or even weakly homotopy equivalent. This shows that sometimes one can get quickly confused when these matters are not clarified first.

Category theory provides a reasonable internal definition of a structure that works well in many contexts, even if it is not always easy to use it in practice. In this sense, the structure foo consists of that what is preserved by the morphisms between foo structures in the category of foo structures and their morphisms. For example, the structure of Boolean algebra, in this sense, consists of all those operations that are preserved by homomorphisms of Boolean algebras. In that sense there is more structure/operations in Boolean algebras than those usually explicitly specified. Note, however, that the objects of a category do not determine the structure in general. For example, the categories of complete lattices and homomorphisms preserving all meets $\bigwedge -Lat$ and of cocomplete lattices and homomorphisms preserving all joins $\bigvee -Lat$ have the same objects (since any partial order has all meets iff it has all joins) but they are not equal, as categories. One can check that they are equivalent and dually equivalent.

The distinction between structure and property is clearer but even here a general formal definition is not present, and, again, for a good reason. The intuitive difference is that a property is like structure but with some kind of uniqueness involved. 'Finiteness' or 'existence of substraction operations' are properties of distributive lattices, whereas 'topology' is an additional structure that one can consider on groups making them into topological groups. In this sense, being a distributive lattice, Heyting algebra, coHeyting algebra or bi-Heyting algebra are all properties of posets. The categorical manifestation of this fact is that the forgetful functors from these categories to $Poset$ (the category of partial orders and monotone maps) are not only faithful but also full on isomorphisms.

These problems with properties and structures become even less clear if the structures are based on categories rather than sets (or posets). One of the reasons is that the uniqueness is usually not expected to be understood as uniqueness 'up to identity' but only 'up to an isomorphism', if we talk about objects and morphisms inside categories, or even 'up to equivalence' if we talk about categories themselves. In particular, if we postulate that 'for every object A there is an object B (a morphism f, a foo α) unique up to an isomorphism such that so and so... , to get an actual function/functor from A's to B's (f's, α') we still need the axiom of choice, possibly even for classes. In usual practice, there are few objections to use it but still mathematicians appreciate very much having elementary definitions of their notions even if they use equivalent notions that are not elementary.

Categorical properties are usually understood as such properties of categories that are invariant under equivalences of categories, i.e., the property P of categories is a categorical property iff whenever the category $\mathcal{A}$ has a property P and category $\mathcal{B}$ is equivalent to $\mathcal{A}$ (notation $\mathcal{A} \simeq \mathcal{B}$), then $\mathcal{B}$ also has property P.

Clearly, no matter which of many definitions of a(n elementary) topos we adopt it is (usually) considered as property of categories, and not an additional structure on categories. The sameness relation is usually[1] understood as equivalence of categories. Thus building theory on the base of one such definition or the other is rather like working in two different linguistic variants of the same theory than like developing two independent (even if related) theories. Changing the notation from one to the other is, from mathematical point of view, even of lesser importance: if we denote terminal object by 7 (usually denoted by 1), binary product by $A \diamondsuit B$ (usually denoted by $A \times B$), power-set object by $Power(A)$ (usually denoted by $\mathbf{P}(A)$), exponential object by $A \times B$ (usually denoted by B^A), subobject classifier by Ξ (usually denoted by Ω), and the generic mono by $maybe : 7 \to \Xi$ (usually denoted by $\mathbf{true} : 1 \to \Omega$), we still have the same notion of an elementary topos. To give an example of additional structure on a category, consider the notion of a monoidal category. If such a monoidal structure exists on a category, it is by no means, in any conceivable sense, unique.

Note, however, that if we have additional structure $\mathcal{M}$ (say bi-Heyting algebra, or monoidal category structure) on the top of some other structure $\mathcal{C}$ (say, partial order or

[1]In some situations coarser equivalences are considered, e.g., in homotopic contexts one can consider homotopy equivalences or weak homotopy equivalences of toposes.

category), then we expect to be able to transfer additional structure $\mathcal{M}$ on $\mathcal{C}$ on the top of an equivalent structure $\mathcal{C}'$ in an essentially unique way appropriate for the context.

Note also that the suitable notion of equivalence is also needed to avoid the proliferation of 'seemingly different' but 'in fact the same' structures.

3 Structures in the thesis

The main structures based on posets considered in the thesis are Heyting, coHeyting and bi-Heyting algebras. The main structure based on categories considered in the thesis is that of elementary topos. Thus in both cases these structures can be thought of as properties of posets and categories, respectively. In the category of posets the natural notion of sameness is an order isomorphism. In the 2-category of categories (functors and natural transformations) the natural notion of sameness is an (adjoint?) equivalence of categories. Note that the notion of isomorphism of categories also makes sense but it is too rare and usually too accidental to be of real interest in practice.

Heyting algebras can be defined as posets that are cartesian closed (as categories) with finite coproducts. When such a definition is adopted, no equations are needed as all the needed equations follow from the fact that these operations are defined via adjunctions, i.e., some universal properties. Similar definition can be given for co- and bi-Heyting algebras.

We have an obvious functor

$$(-)^\circ : \mathit{Hey} \longrightarrow \mathit{coHey}$$

that reverses the order in the algebras (and changes operations accordingly) but keeps the universes and morphisms unchanged. It fits the following commuting square

$$\begin{array}{ccc} \mathit{Hey} & \xrightarrow{(-)^\circ} & \mathit{coHey} \\ {\scriptstyle U_H}\downarrow & & \downarrow{\scriptstyle U_{cH}} \\ \mathit{Poset} & \xrightarrow[(-)^\circ]{} & \mathit{Poset} \end{array}$$

where U_H and U_{cH} associates with algebras their ordered universes (defined from operations in a standard way). Again the functor $(-)^\circ : \mathit{Poset} \longrightarrow \mathit{Poset}$ reverses the order leaving the universes of posets fixed.

The fact that the notions of being Heyting or co-Heyting algebra are properties of partial orders can be formally justified by saying that the forgetful functors U_H and U_{cH} are full on isomorphisms, in addition to being faithful (even if they are not full).

Definitions 7 and 8 on pages 82-83 of elementary toposes are different in flavour. These two definitions well reflect the overall differences in the two most popular kinds of definitions of elementary toposes (there is a plethora of different kinds of definitions of an elementary topos; some of them are listed in the introductions to 'Skeches of an Elephant' A Topos Compendium of P.T. Johnstone). Definition 7 is not elementary, it talks about isomorphisms of a class of functors. Here one can ask whether these isomorphisms constitute a part of the structure or not. Such isomorphisms are not unique, however they are unique up to a unique isomorphism of the representable objects. Thus if

$\varphi : Sub(-) \to Hom(-, \Omega)$ and $\varphi' : Sub(-) \to Hom(-, \Omega')$ are two natural isomorphisms of functors from a category $\mathcal{E}$ to Set, by Yoneda Lemma, there is a unique isomorphism $f : \Omega \to \Omega'$ so that

$$Hom(-, \Omega) \circ \varphi = \varphi'.$$

If a subobject classifier has no non-trivial automorphisms, such an isomorphism is unique once the subobject classifier is fixed.

On the other hand, the definition 8 is elementary. It quantifies over objects and morphisms of the topos but does not postulate any large structures like isomorphism of functors on a large category. As we already indicated, the proof of equivalence of these definitions needs the axiom of choice for classes. Note also that the above definitions of a topos do not need any exactness properties (=equations) to be specified explicitly. In particular, all those exactness properties that allow to give a sound interpretation of Intuitionistic logic follows.

4 Intuitionistic and Paraconsistent Logics

Chapters 1 and 2 of the thesis constitute a gentle introduction to the Intuitionistic logic and Paraconsistent logics with a lot of pointers to literature to read more about them. It is not the place one can really learn much of the technical issues involved. The subject is way too broad to be summarized within a single thesis. Intuitionistic logic has a vast literature so it is not an easy task to introduce it better. The Paraconsistent logics are less known and here the introduction presented in the thesis is much more informative and useful.

5 The main content of the thesis

Chapter 3 contains the main part of the thesis. Section 3.1 introduces in a formal way Heyting, co-Heyting, and bi-Heyting algebras. co-Heyting algebras constitute a candidate for the algebraic counterpart of a well behaved paraconsistent logic. The semantics is given in topological spaces only. The issues concerning soundness and completeness are not mentioned. It is to some regret that there are only few indications (lattices of subobjects of some toposes) that co-Heyting and even bi-Heyting algebras can occur in a natural way. For example, any finite or free Heyting algebra is a bi-Heyting algebra.

Section 3.2 provides a motivation and introduction to the definition of a cotopos. It concentrates on the definition of the complement-classifier that is visibly nothing but a subobject classifier to a mathematically minded reader.

Section 3.3 is analyzing the kind of uniqueness of the generic subobject in a topos. I understand that here philosophical attitudes with respect to unique entities but unique only up to some (unique!) transformation, isomorphism in this case, might be different from the mathematical ones and worth deeper analysis. For a mathematically minded person the notion of topos and cotopos as presented in the thesis is the same notion, i.e., these definitions even do not require a more sophisticated analysis resulting in conclusions that they are linguistic variants of one another. However, it is interesting for a mathematician like me to see how philosophers approach the problem. The argument presented in the thesis goes through another definition 7 that is only equivalent to definition 8 if we accept the axiom of choice for classes. Sections 3.4 provides a similar analysis as the one in Section 3.3 but applied to the power object. The Chapter 3 ends with a simple but non-trivial example of a subobject classifier in the category $Set^{\rightarrow}$.

Thus I agree with the main claim of the thesis that C. Mortensen's definition of a cotopos that is supposed to provide the semantics for paraconsistent logics agrees with the definition(s) of an elementary topos. Elementary toposes provide a sound (and complete) semantics for intuitionistic higher order logic. It is interesting whether they can still provide a semantics for some kind of paraconsistent logics. There are several problems to overcome. One of them, mentioned in the thesis, is that the substitution operation does not preserve the coHeyting operation of pseudo-difference, in general.

6 Style of writing

The thesis is generally clearly written. Expository parts (Chapters 1 and 2) are very well chosen to fit the needs of thesis.

7 Conclusion

I think that the thesis of doctor Mariusz Stopa presents a philosophical new perspective on classical theory of elementary toposes. Therefore, as I wrote at the beginning of this report, I think that the thesis meets all the requirements for the doctoral thesis (Dz. U. z 2016 r. poz. 882) and therefore I recommend that the thesis of doctor Mariusz Stopa be accepted to be defended by the candidate in front of a Jury.

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